## UNIVERSITY *of* LIMERICK

O L L S C O I L   L U I M N I G H

## Faculty of Science and Engineering
### Department of Mathematics & Statistics

## Special Mathematics Entrance Examination
## Higher Level

**DATE:**   Thursday 21st August 2014

**TIME:**   14.30-17.30 (3 HOURS)

### INSTRUCTIONS TO CANDIDATES:

There are **two** sections in this examination paper.

Section A:   6 questions,   25 marks each.
Section B:   3 questions,   50 marks each.

### ANSWER ALL QUESTIONS

The invigilator will provide answer books, graph paper and a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

You will lose marks if all necessary work is not clearly shown.

Answers should include appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

## Section A (6 Questions, 25 marks each)

1. Two fair six-sided dice are tossed.

    (a) Show the resulting sample space in a diagram.

    (b) If $x$ is the sum of the scores on the two dice, find

       (i) $P(x = 6)$,

       (ii) $P(x > 6)$,

       (iii) $P(x = 7|x > 5)$.

    (c) Niamh plays a game where she tosses two dice.
        If the sum is 6, she wins €3.
        If the sum is greater than 6, she wins €1.
        If the sum is less than 6, she loses $p$ euros.
        Find the value of $p$ for which Niamh neither wins or loses.

2. Let $z_1 = 1 - 2i$, where $i^2 = -1$.

    (a) $z_1$ is the root of the equation

    $$2z^3 - 7z^2 + 16z - 15 = 0.$$

    Find the other two roots of the equation.

    (b)  (i) Let $w = z_1\bar{z}_1$, where $\bar{z}_1$ is the conjugate of $z_1$.
             Plot $z_1$, $\bar{z}_1$ and $w$ on an Argand diagram and label each point.

         (ii) Find the angle $\angle\bar{z}_1 w z_1$, correct to the nearest degree.

3.  (a) The seating in a small lecture has curved rows of seats. As a result, the number of seats per row increases by two as the rows go from front to back.

    (i) If there are $12$ seats in the front row, how many seats are there in the $7^{th}$ row?

    (ii) Given that the back row is the $21^{st}$, what is the seating capacity of the lecture theatre?

    (b) In a geometric series, $T_1 = x$, $T_2 = x^2 - 2x$, $x \neq 0$.

    (i) Write down the common ratio of the series in terms of $x$.

    (ii) If the sum to infinity of the series is $5$, find the value of $x$.

    (iii) Find the range of values of $x \in \mathbb{R}$ for which the series has a sum to infinity.

4.  (a) Differentiate $\dfrac{x}{1 - x^2}$ with respect to $x$.

    (b) Given $f(x) = \ln 3x - 3x$, $x > 0$.

    Show that $\left(\frac{1}{3}, -1\right)$ is a local maximum point of $f(x)$.

    (c) Using integration, find the area enclosed by the parabola $y^2 = 2(x+2)$ and the $y$-axis.

5. The points $A(-4, 6)$, and $C(-1, 1)$ are the opposite vertices of a parallelogram $ABCD$.
   The sides $BC$ and $CD$ of the parallelogram lie along the lines $x + 3y - 2 = 0$ and $x - y + 2 = 0$ respectively.

   (i) Find the coordinates of the vertex $D$.

   (ii) Calculate the size of the acute angle between the diagonals to the nearest degree.

6. The equation of a circle is $x^2 + y^2 - 4x + 2y - 20 = 0$.

   (i) Write down the coordinates of the centre of the circle and the length of its radius.

   (ii) Verify that $P(-1, 3)$ is a point on the circle.

   (iii) The tangent, $T$, to the circle at $P$ cuts the $x$-axis at $Q$.
   Find the coordinates of $Q$.

   (iv) $L$ is the line through $Q$ perpendicular to $T$. Verify that $L$ is also a tangent to the circle.

## Section B (3 Questions, 50 marks each)

7. (a) A farmer buys $1,200$m of fencing with the intention of fencing off a rectangular field that borders a straight river. There is no need to fence along the river.

   (i) What are the dimensions of the field that will give the largest area?

   (ii) Calculate the maximum area.

   (b) A man launches his boat from point $A$ on a bank of a straight river, $3$ km wide, and wants to reach point $B$, $8$ km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point $C$ and then run to $B$, or he could row directly to $B$, or he could row to some point $D$ between $C$ and $B$ and then run to $B$.

   If he can row at $6$ km/h and run at $8$ km/h, where should he land to reach $B$ as soon as possible? (Assume that the speed of the water is negligible compared with the speed at which the man rows.)

8.  A researcher wants to investigate whether the length of time people spend in education affects the income they earn. In a survey of twelve adults, they are asked to state their annual income and the number of years they spent in full-time education. The resultant data are given in the table below.

| Years of Education | 11 | 12 | 13 | 13 | 14 | 15 | 16 | 16 | 17 | 17 | 17 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Income (€1,000) | 28 | 30 | 35 | 43 | 55 | 38 | 45 | 38 | 55 | 60 | 30 | 58 |

   (i)  Identify the dependent and independent variables.

   (ii)  Represent the data in a scatter plot, showing the relevant axes.

   (iii)  Calculate the coefficient of linear correlation.

   (iv)  What can you conclude from your results in parts (ii) and (iii)?

   (v)  Add the line of best fit to the completed scatter plot in (ii).

   (vi)  Use the line of best fit to estimate the annual income of a person with 14 years full-time education.

   (vii)  By taking suitable readings from your diagram, or otherwise, calculate the slope of the line of best fit.

   (viii)  Explain how to interpret this slope in this context.

9. The arch of a bridge is in the shape of a parabola, as shown. The length of the span of the arch, $[AB]$, is $48$ metres.
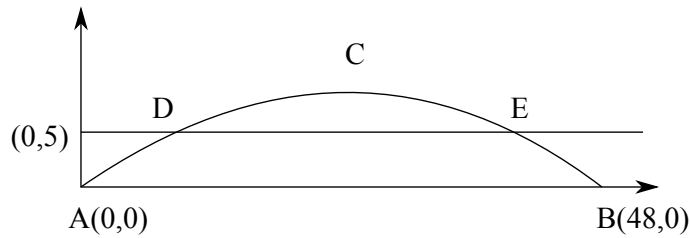


(a) Using the coordinate plane, with $A(0,0)$ and $B(48,0)$, the equation of the parabola is $y = -0.013x^2 + 0.624x$.
Find the coordinates of $C$, the highest point of the arch.

(b) The perpendicular distance between the walking deck, $[DE]$, and $[AB]$ is $5$ metres.
Find the coordinates of $D$ and of $E$. Give your answers correct to the nearest whole number.

(c) Using integration, find the area of the shaded region, $ABED$, shown in the diagram below.
Give your answer correct to the nearest whole number.
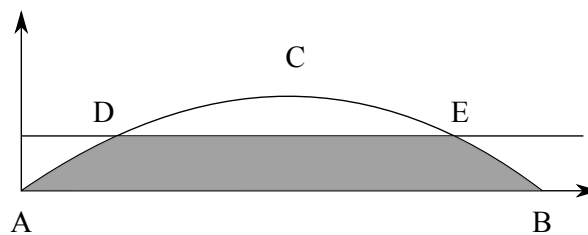


(d) Write the equation of the parabola in part (a) in the form $y - k = p(x - h)^2$, where $k$, $p$ and $h$ are constants.

(e) Using what you learned in part (d) above, or otherwise, write down the equation of a parabola for which the coefficient of $x^2$ is $-2$ and the coordinates of the maximum point are $(3, -4)$.